October Math Gems

Problem of the week 11

§1 Problems

Problem 1.1. Let n be a positive integer s.t

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}=\frac{1}{9}$$

find the value of n.

Solution.

$$\sum_{n=9}^{n}\frac{1}{n\sqrt{(n+2)}+(n+2)\sqrt{n}}=\sum_{n=9}^{n}\frac{1}{\sqrt{n(n+2)}(\sqrt{n}+\sqrt{n+2})}=$$

$$\sum_{n=9}^{n}\frac{1}{\sqrt{n(n+2)}(\sqrt{n}+\sqrt{n+2})}\times\frac{\sqrt{n}-\sqrt{n+2}}{\sqrt{n}-\sqrt{n+2}}=\sum_{n=9}^{n}\frac{\sqrt{n}-\sqrt{n+2}}{-2\sqrt{n(n+2)}}=\frac{1}{-2}\sum_{n=9}^{n}\frac{\sqrt{n}-\sqrt{n+2}}{\sqrt{n(n+2)}}=$$

$$\frac{1}{-2}\sum_{n=9}^{n}\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n}}$$

Let's plug numbers:

$$\text{for } n=9,\quad \frac{1}{\sqrt{11}}-\frac{1}{\sqrt{9}}$$

$$\text{for } n=10,\quad \frac{1}{\sqrt{12}}-\frac{1}{\sqrt{10}}$$

$$\text{for } n=11,\quad \frac{1}{\sqrt{13}}-\frac{1}{\sqrt{11}}$$

$$\text{for } n=12,\quad \frac{1}{\sqrt{14}}-\frac{1}{\sqrt{12}}$$

$$\text{for } n=13,\quad \frac{1}{\sqrt{15}}-\frac{1}{\sqrt{13}}$$

and so on to get

$$\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n-2}}$$

$$\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n-1}}$$

$$\frac{1}{\sqrt{n+2}}-\frac{1}{\sqrt{n}}$$

we can see that all terms will be canceled when we add these terms together except $-\frac{1}{\sqrt{9}}$ and $\frac{1}{\sqrt{n+2}}$.

$$\frac{1}{-2} \times \left(-\frac{1}{\sqrt{9}} + \frac{1}{\sqrt{n+2}}\right) = \frac{1}{9} \implies \frac{1}{\sqrt{n+2}} = \frac{1}{9}$$

$$n + 2 = 81 \implies n = 79$$

□

Problem 1.2. What is the remainder when 9^{1190} is divided by 11?

Solution. Using Fermat's little theorem:
If the $gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \mod 11$

$$9^{1190} = 9^{119\times10} = (9^{119})^{10} \equiv 1 \mod 11$$

As, $10 = 11–1$. □

Problem 1.3. prove that $7|(2222^{5555} + 5555^{2222})$.

Solution.

$$2222 = 101 \times 22 \quad , \quad 5555 = 55 \times 101$$

$$2222 = 101 \times 22 \equiv 3 \times 1 \mod 7$$

$$5555 = 101 \times 55 \equiv 3 \times -1 \mod 7$$

$$(2222^{5555} + 5555^{2222}) \equiv ((3)^{5555} + (-3)^{2222}) \equiv ((-4)^{5555} + (4)^{2222}) \mod 7$$

As,

$$2^6 \equiv 1 \mod 7$$

$$((-4)^{5555} + (4)^{2222}) = 4^{2222}(1 - 4^{3333}) = 4^{2222}(1 - 2^{2\times3333}) =$$

$$4^{2222}(1 - 2^{6\times1111}) \equiv 4^{2222}(1 - 1) \equiv 0 \mod 7$$

Hence we proved it! □

Problem 1.4. When a positive integer n is divided by $5, 7, 9, 11$, the remainders are $1, 2, 3, 4$ respectively. Find the minimum value of n.

Solution. The difference between the divisors and the remainders is not equal. So, in the next step, we will make the difference between the remainders and dividers is equal

$$5 - (1 \times 2) = 3$$

$$7 - (2 \times 2) = 3$$

$$9 - (3 \times 2) = 3$$

$$11 - (4 \times 2) = 3$$

$LCM = \frac{5\times7\times9\times11}{gcd(5,7,9,11)} = 3465$, so the least number n is

$$\frac{3465 - 3}{2} = 1731$$

□

Problem 1.5. Prove that $7^n - 1$is always divisible by 6.

Solution. Using mathematical induction.
consider Base case $(n = 1) \implies 7^1 - 1 = 6$ which is divisible by 6 (base case is true),
Induction step, suppose that

$$7^k - 1$$

is always divisible by 6.
Now, we want to prove that $7^{k+1} - 1$ is always divisible by 6,

$$7^{k+1} - 1 = 7 \times 7^k - 1 = 7(7^k - 1) + 6$$

As we assumed before $7^k - 1$ is always divisible by 6. So, the two sides are divisible by 6 which is ($7^k - 1, 6$).
The induction step is completed. $\square$

Problem 1.6. Let N be a positive for which the sum of its smallest factors is 4 and the sum of the largest factors is 204. Find the value of $\frac{N}{3}$.

Solution. The smallest factors will be 1 and 3 and the largest two factors will be N and $\frac{N}{3}$. So,

$$N + \frac{N}{3} = 204 \implies N = 153 \quad \text{and} \quad \frac{N}{3} = 51$$

$\square$

Problem 1.7. Let a, b are relatively prime integers with $a > b > 0$ and

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}$$

What is the value of $a - b$?

Solution.

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a-b)^2} = \frac{(a-b)(a^2 + ab + b^2}{(a-b)(a^2 - 2ab + b^2)} = \frac{(a^2 + ab + b^2)}{(a^2 - 2ab + b^2)} = \frac{73}{3}$$

Let $a^2 + b^2 = x$ and $ab = y$. So,

$$\frac{x + y}{x - 2y} = \frac{73}{3} \implies \frac{x}{y} = \frac{149}{70}$$

as a, b are relatively prime. So,

$$x = 149, \quad y = 70 \implies a^2 + b^2 = 149, \quad ab = 70$$

$$a > b > 0 \implies a = 10, \quad b = 7$$

$$a - b = 10 - 7 = 3$$

$\square$

Problem 1.8. A positive integer n is nice if a positive integer m was exactly four positive divisors (including 1 and m) s.t the sum of four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \ldots, 2019\}$ are nice?

Solution. The only number in the set is 2016 (a nice number). For this nice number, the following value of m has exactly four divisors such that the sum of the four divisors of each is 2016; namely

$$1509, 1757, 1837, 1909, \text{ and } 1927$$

□

Problem 1.9. Find the sum of digits of the largest possible integer n such that $n!$ ends with exactly 100 zeros.

Solution. 100! ends with exactly

$$\lfloor\frac{100}{5}\rfloor + \lfloor\frac{100}{25}\rfloor = 20 + 4 = 24$$

So, for getting the maximum number that ends with 100 zeros it must be greater than 400!

$$\lfloor\frac{400}{5}\rfloor + \lfloor\frac{400}{25}\rfloor + \lfloor\frac{400}{125}\rfloor = 99$$

So, trying numbers after 400 to get the maximum number ends with 100 zeros. Let's trying 409!

$$\lfloor\frac{409}{5}\rfloor + \lfloor\frac{409}{25}\rfloor + \lfloor\frac{409}{125}\rfloor = 100$$

So the sum of digits of 409 is 4 + 9 + 0 = 13 □

Problem 1.10. The number of numbers of the form $30a0b03$ that are divisible by 13 where a, b are digits is?

Solution.

$$30a0b03 = 3000003 + 10000a + 100b = (13q_1 + 6) + (13q_2 + 3a) + (13q_3 + 9b) =$$

$$13(q_1 + q_2 + q_3) + 3(2 + a + 3b)$$

So $(2 + a + 3b)$ should be a multiple of 13.

$$(2 + a + 3b) = 13 \rightarrow\ 1^{st} \text{ case}$$

$$(2 + a + 3b) = 26 \rightarrow 2^{nd} \text{ case}$$

For the 1^{st} case ,

$$a + 3b = 11$$

the possibilities for (a, b) are

$$(8, 1), (5, 2), (2, 3)$$

For the 2^{nd} case ,

$$a + 3b = 24$$

the possibilities for (a, b) are

$$(9, 5), (6, 6), (3, 7), (0, 8)$$

so the total numbers of this form $30a0b03$ are 7. □

Problem 1.11. The digits $1, 2, 3, 4, 5, 6, 7,$ and 9 are used to form four two-digit prime numbers, with each digit used exactly once. Then the sum of these four primes is given by $10A$. Find the value of A.

Solution. The four digits will be odd prime numbers and their sum will be $10A$ so $1, 3, 7, 9$ will be in units. When 1 will be in units there are two possibilities which are 31 and 41 and as 3 will be in units in another number (and as we mentioned in the question each number will not be repeated) so the only number is 41. we can conclude the other three numbers. They will be

$$23, 67, 59$$

The sum:

$$41 + 23 + 67 + 59 = 190 = 10A \implies A = 19$$

$\square$

Problem 1.12. Find the number of positive divisors of

$$(2008^3 + 3 \times 2008 \times 2009 + 1)^2$$

Solution. Suppose that $a = 2008$,

$$a^3 + 3 \times a \times (a + 1) + 1 = a^3 + 3 \times (a^2 + a) + 1 = a^3 + 3a^2 + 3a + 1 = (a + 1)^3$$

$$2008^3 + 3 \times 2008 \times 2009 + 1 = (2009)^3$$

$$2009 = 7^2 \times 41$$

$$(2008^3 + 3 \times 2008 \times 2009 + 1)^2 = (2009)^6 = (7^2 \times 41)^6 = 7^{12} \times 41^6$$

Number of divisors are $(\tau) = (6 + 1)(12 + 1) = 91$ $\square$

Problem 1.13. Prove that the equation $a^2 + b^2 - 8c = 6$ has no integer solution.

Solution. We will work on $\mod 8$. As we know that

$$a^2 \equiv 1, 4, 0 \mod 8$$

$$a^2 + b^2 \equiv 0, 1, 2, 5 \mod 8$$

$$8c + 6 \equiv 6 \mod 8$$

so it is a contradiction. Hence there is no integer solution. $\square$

Problem 1.14. There are N numbers of positive integers not exceeding 2001 and are multiples of 3 or 4 but not 5. Then find the value of

$$\frac{N - 1}{50}$$

Solution. Multiples of 3,

$$\lfloor \frac{2001}{3} \rfloor = 667$$

Multiples of 4,

$$\lfloor \frac{2001}{4} \rfloor = 500$$

Multiples of 12,

$$\lfloor \frac{2001}{12} \rfloor = 166$$

Multiples of 60,

$$\lfloor \frac{2001}{60} \rfloor = 33$$

Multiples of 15,

$$\lfloor \frac{2001}{15} \rfloor = 133$$

Multiples of 20,

$$\lfloor \frac{2001}{20} \rfloor = 100$$

Then, multiples of 3 or 4 but not 5

$$N = 667 + 500 + 33 - (166 + 133 + 100) = 801$$

$$\frac{N-1}{50} = \frac{801-1}{50} = 16$$

$\square$

Problem 1.15. Let n be a 5-digit number, and let r and q be a quotient and remainder, respectively, when n is divided by 100. Let the total number of value of n for which $q+r$ divisible by 11 is P. Then find the last two digits of P.

Solution. As $11|(q+r)$ then $11|(99q+q+r) \implies 11|(100q+r)$ which is n. So,

$$11|n$$

the total possible values for 5-digit numbers is

$$\frac{99990 - 10010}{11} + 1 = 8181$$

The last two digits of p is 81. $\square$

Problem 1.16. Let x and y be a two-digit integers such that y is obtained by reversing the digits of x. Suppose that the integers x and y satisfy $x^2 - y^2 = m^2$ for positive integer m. What is the value of

$$\frac{x+y+m}{11}$$

?

Solution.

$$x = 10a + b \implies y = 10b + a$$

In the question $x^2 - y^2 = m^2$,

$$(10a+b)^2 - (10b+a)^2 = m^2$$

$$(10a + b + 10b + a)(10a + b - 10b - a) = m^2$$

$$(11a + 11b)(9a - 9b) = m^2 \implies 99(a+b)(a-b) = m^2$$

As, a, b number from 1 to 9 so the only possibility for making m a perfect square is $(a+b) = 11$ and $(a-b) = 1$,

$$99(a+b)(a-b) = 33 \times 3 \times 11 \times 11 = 33 \times 33 = 33^2 = m^2 \implies m = 33$$

$$\begin{cases} a + b = 11 \\ a - b = 1 \end{cases} \implies a = 6, \quad b = 5$$

$$x = 65, \quad y = 56, \quad m = 33 \implies \frac{x+y+m}{11} = \frac{65+56+33}{11} = 14$$

$\square$

Problem 1.17. How many positive integer n does $1+2+3+4+\cdots+n$ evenly divide from $6n$?

Solution.

$$1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$$

$$\frac{6n}{\frac{n(n+1)}{2}}=\frac{12}{n+1}=\text{integer}$$

So, $n+1$ must be a factor of 12. The factors of 12:

$$1,12,2,6,3,4$$

$$n+1=1 \implies n=0$$

$$n+1=12 \implies n=11$$

$$n+1=2 \implies n=1$$

$$n+1=6 \implies n=5$$

$$n+1=3 \implies n=2$$

$$n+1=4 \implies n=3$$

Hence 0 is non-positive and non-negative number, so there are only five possibility of n which are $\{11,1,5,2,3\}$ □

Problem 1.18. If N is the number of four-digit positive integers with at least one digit that is a 2 or a 3 then find the sum of the digits in N.

Solution. Number of the four-digit positive integers is

$$9\times 10\times 10\times 10=9000$$

Number of the four-digit positive integers that do not have either 2 or 3

$$7\times 8^3=3584$$

the number of four-digit positive integers with at least one digit that is a 2 or a 3,

$$N=9000-3584=5416$$

The sum of digits of N is $5+4+1+6=16$ □

Problem 1.19. What is the sum of the digits of the square 111 111 111 ?

Solution. As, $11^2=121$
$111^2=12321$
So, The sum of the digits of 111111111^2 is

$$\frac{8\times 9+9\times 10}{2}=81$$

□

Problem 1.20. If a, b, c are positive real numbers such that

$$ab + a + b = bc + b + c = ca + a + c = 35$$

Then the value of $(a+1)(b+1)(c+1) = x^y$. Find the product of x and y.

Solution.

$$(a+1)(b+1) = (ab + a + b + 1) = (35 + 1) = 36$$
$$(b+1)(c+1) = (bc + b + c + 1) = (35 + 1) = 36$$
$$(a+1)(c+1) = (ac + a + c + 1) = (35 + 1) = 36$$

As,

$$(a+1)(b+1)(b+1)(c+1)(a+1)(c+1) = (a+1)^2(b+1)^2(c+1)^2 = 36^3$$

$$(a+1)(b+1)(c+1) = \sqrt{36^3} = 216$$
$$x^y = 216 = 6^3 \implies x \times y = 3 \times 6 = 18$$

$\square$
